University of California, Berkeley
Physics 110B, Fall 2004 (*Strovink*)
PROBLEM SET 13
Due at 5 PM on Wednesday, December 1, 2004

Problems 66-72 develop scalar diffraction theory and apply it to Fraunhofer diffraction, which reduces to a Fourier transform.

66. Green's theorem.
Denote by $\vec{G}$ a vector field, and start from the *divergence theorem*

$$\int \nabla \cdot \vec{G}\, d\tau = \oint \vec{G} \cdot \hat{n}\, da\ ,$$

where $\hat{n}$ is the (outward) direction of the surface area element $d\vec{a}$, and the left-hand integral extends over the volume enclosed by the right-hand surface.
(a.)
Substituting $\vec{G} = V\nabla U$, where V and U are scalar fields, show that

$$\int \big(\nabla V \cdot \nabla U + V\nabla^2 U\big)\, d\tau = \oint V\frac{\partial U}{\partial n}\, da\ .$$

(b.)
Show that

$$\int \big(V\nabla^2 U - U\nabla^2 V\big)\, d\tau = \oint \Big(V\frac{\partial U}{\partial n} - U\frac{\partial V}{\partial n}\Big)\, da\ .$$

(c.)
If V and U both satisfy the scalar Helmholtz equation,

$$\big(\nabla^2 + k^2\big)(U, V) = 0\ ,$$

where k is a constant, show that

$$0 = \oint \Big(V\frac{\partial U}{\partial n} - U\frac{\partial V}{\partial n}\Big)\, da\ .$$

This is *Green's theorem for solutions to the scalar Helmholtz equation*.

67. Fresnel-Kirchoff integral theorem.
Please use the notation and results of the previous problem.
(a.)
Consider a closed surface consisting of an inner sphere of radius R, centered at the origin, and an arbitrary closed outer surface $\mathcal{A}$. Apply the result of part (c.) to the combined surface. Take V to be an inward-propagating spherical wave

$$V = V_0\frac{e^{i(kr+\omega t)}}{r}\ .$$

In the limit $R \to 0$, show that

$$U(0) = \tfrac{1}{4\pi}\oint\Big(\frac{e^{ikr}}{r}\frac{\partial U}{\partial n} - U\frac{\partial}{\partial n}\big(\frac{e^{ikr}}{r}\big)\Big)\, da\ ,$$

where the integral is taken only over $\mathcal{A}$. This is the *Kirchoff integral theorem*.
(b.)
Now punch a hole ("aperture") in $\mathcal{A}$. Place a point source S outside $\mathcal{A}$; the origin (now called "observation point P") still lies inside $\mathcal{A}$. The source radiates an outward-propagating scalar spherical wave

$$U = U_0\frac{e^{i(kr'-\omega t)}}{r'}\ ,$$

where $\vec{r}\,'$ is a vector from S to a point in space. Using the result of (a.), assume that the opacity of the remainder of $\mathcal{A}$ allows the integral to be carried out over only the aperture ("ap"). In the far zone limit $kr', kr \gg 1$, show that

$$U_P = \frac{-ikU_0e^{-i\omega t}}{4\pi}\int_{\text{ap}}\frac{e^{ik(r+r')}}{rr'}\big(\hat{r}\cdot\hat{n} - \hat{r}'\cdot\hat{n}\big)\, da\ ,$$

where $\vec{r}$ ($\vec{r}\,'$) is a vector from P (point S) to a point on the element of aperture da, and $\hat{n}$ is the (outward from P) normal to da. This is the *Fresnel-Kirchoff integral theorem*; it is the starting point for the study of diffraction in the *scalar field approximation*.

68. Knife-edge diffraction.
A plane wave of initial irradiance I_0 propagating along $\hat{z}$ is incident upon a semi-infinite totally absorbing screen lying in the $z = 0$ plane. The screen extends from $-\infty < x < \infty$ and $-\infty < y < 0$. An observer stationed at $(0, 0, z)$, where $kz \gg 1$, detects an irradiance I'. What is I'/I_0, and why?

69. Fourier diffraction.
The *convolution* of two functions $f(x)$ and $g(x)$, denoted by $(f \otimes g)(x)$, is defined by

$$f \otimes g \equiv \int_{-\infty}^{\infty} dx' \, f(x') \, g(x - x') \ .$$

Define the *Fourier transform* $\mathcal{F}_\mu\big(g(x)\big)$ by

$$\mathcal{F}_\mu\big(g(x)\big) \equiv \int_{-\infty}^{\infty} dx \, g(x) \, e^{-i\mu x} \ .$$

(a.)
As a warmup, prove that

$$f \otimes g = g \otimes f \ .$$

(b.)
For use in part (d.), prove that

$$\mathcal{F}_\mu\big(f(x) \otimes g(x)\big) = \mathcal{F}_\mu\big(f(x)\big) \, \mathcal{F}_\mu\big(g(x)\big) \ .$$

(c.)
If $f(x)$ is the aperture function for a pair of thin slits separated by d,

$$f(x) \propto \delta(x - \tfrac{d}{2}) + \delta(x + \tfrac{d}{2}) \ ,$$

and if $g(x)$ is the aperture function of a single slit of thickness a,

$$g(x) \propto \theta(x + \tfrac{a}{2}) - \theta(x - \tfrac{a}{2}) \ ,$$

show that $f \otimes g$ is the aperture function corresponding to two slits of thickness a, separated (centerline-to-centerline) by d.
(d.)
In the Fraunhofer approximation, where $\vec{r}\,'$ and $\vec{r}$ (cf. Problem 67) are paraxial and the wavefront curvature across the aperture is negligible, the scalar "optical disturbance" amplitude is

$$U_P(\mu, \nu) \propto \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, g(x, y) e^{-i(\mu x + \nu y)} \ ,$$

where U_P is measured at the transform plane (X, Y), the aperture function g is measured at the aperture plane (x, y), μ and ν are defined by

$$\mu \equiv \frac{kX}{f} \quad \nu \equiv \frac{kY}{f} \ ,$$

and f is the focal length of the thin field lens located an equidistance f from the aperture and transform planes. Write down the diffraction pattern

$$\frac{I_N(\psi_x, \psi_y)}{I_1(0, 0)}$$

for N slits of center-to-center separation $\Delta x = d$ and thicknesses $\delta x = a$ and $\delta y = b$, where

$$(\sin)\psi_x \equiv \frac{X}{f}$$
$$(\sin)\psi_y \equiv \frac{Y}{f} \ .$$

You may use the fact – directly obtainable by applying the Fourier transform – that

$$\frac{I_N(\psi_x)}{I_1(0)} = N^2 \frac{\sin^2\left(\frac{Nkd}{2} \sin\psi_x\right)}{\left(N \sin\left(\frac{kd}{2} \sin\psi_x\right)\right)^2}$$

for N thin slits of infinite length and separation d, and that

$$\frac{I(\psi_x)}{I(0)} = \mathrm{sinc}^2\left(\tfrac{ka}{2} \sin\psi_x\right)$$

for a single slit of infinite length and thickness a.

70. Quadruple slit.
Consider four equally spaced long ($\Delta y = \infty$) thin slits, located at $x = \pm\frac{d}{2}$ and $x = \pm\frac{3d}{2}$. As usual, $\tan\psi_x = \frac{dx}{dz}$ of the outgoing wavefront.
(a.)
Write down the standard result

$$\mathcal{R}(\psi_x) \equiv \frac{I(\psi_x)}{I(\psi_x = 0)}$$

for the Fraunhofer diffraction pattern from $N = 4$ equally spaced thin slits.
(b.)
Consider the full diffracted *amplitude* to be the superposition of the diffracted *amplitudes* from a pair of slits at $x = \pm\frac{d}{2}$ and a pair of slits at $x = \pm\frac{3d}{2}$. Write down $\mathcal{R}(\psi_x)$ as a quantity proportional to the modulus^2 of the sum of the diffracted amplitudes from the two pairs of slits.
(c.)
Consider the aperture function for these four slits to be the convolution of a pair of δ-functions separated by d and another pair of δ-functions separated by $2d$ (both pairs are symmetric about $x = 0$). Write down $\mathcal{R}(\psi_x)$ as the product of two two-slit $\mathcal{R}$'s.
(d.)
Are your answers to parts (a.), (b.), and (c.) equivalent? Why or why not?

71. Fuzzy thick slit.
Please use the notation and results of the previous problem. Consider a trapezoidal aperture function

$$\begin{aligned} g(x) &= 1 \qquad |x| < \tfrac{a}{2} \\ &= 0 \qquad |x| > a \\ &= \tfrac{2}{a}(x+a) \qquad -a < x < -\tfrac{a}{2} \\ &= \tfrac{2}{a}(a-x) \qquad \tfrac{a}{2} < x < a\,. \end{aligned}$$

Fraunhofer conditions apply. Under these conditions, calculate the slit's diffraction pattern

$$\mathcal{R}(\psi_x) \equiv \frac{I(\psi_x)}{I(\psi_x = 0)}\,.$$

72. Thick slits with wave plates.
A linearly ($\hat{x}$) polarized plane EM wave traveling along $\hat{z}$ is incident on an opaque baffle located in the plane $z = 0$. The baffle has two slits cut in it, which are of infinite extent in the $\hat{y}$ direction. In the $\hat{x}$ direction, the slit widths are each a and their center-to-center distance is d. (Obviously $d > a$, but you may *not* assume that $d \gg a$.) The top and bottom slits are each an equal distance from $x = 0$.

The diffracted image is viewed on a screen located in the plane $z = L$, where $L \gg d$; also $\lambda L \gg d^2$, where λ is the EM wavelength.

Quarter-wave plates are placed in each slit. They are identical, except that the top plate's "slow" (high-index) axis is along $(\hat{x}+\hat{y})/\sqrt{2}$ ($+45°$ with respect to the $\hat{x}$ axis), while the bottom plate's slow axis is along $(\hat{x}-\hat{y})/\sqrt{2}$ ($-45°$ with respect to the $\hat{x}$ axis).
(a.)
What is the state of polarization of the diffracted light that hits the center of the screen, at $x = y = 0$? Explain.
(b.)
At what diffracted angle ψ_x does the first minimum of the irradiance occur?